4

TERMINOLOGY

Argand plane
argument
complex arithmetic
complex conjugate
De Moivre's theorem
locus
polar form

COMPLEX NUMBERS

THE COMPLEX PLANE

4.01 Addition and subtraction in the complex plane
4.02 Multiplication in the complex plane
4.03 Division in the complex plane
4.04 Complex number operations as transformations
4.05 Graphing subsets of the complex plane: subsets involving modulus
4.06 Graphing subsets of the complex plane: subsets not involving modulus
Chapter summary
Chapter review

Prior learning

COMPLEX ARITHMETIC USING POLAR FORM

- define and use multiplication, division, and powers of complex numbers in polar form and the geometric interpretation of these (ACMSM082)
- prove and use De Moivre's theorem for integral powers (ACMSM083)

THE COMPLEX PLANE (THE ARGAND PLANE)

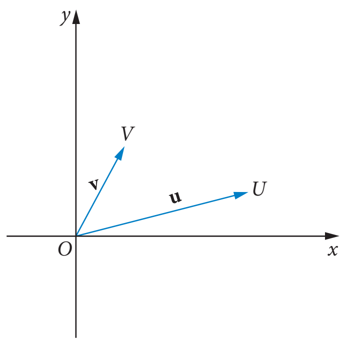
- examine and use addition of complex numbers as vector addition in the complex plane (ACMSM084)
- examine and use multiplication as a linear transformation in the complex plane (ACMSM085)
- identify subsets of the complex plane determined by relations such as $|z - 3i| \leq 4$, $\frac{\pi}{4} \leq \text{Arg}(z) \leq \frac{3\pi}{4}$, $\text{Re}(z) > \text{Im}(z)$, and $|z - 1| = 2|z - i|$ (ACMSM086)

4.01 ADDITION AND SUBTRACTION IN THE COMPLEX PLANE

Chapter 2 was about the four operations with complex numbers and De Moivre's theorem. This chapter is about geometric interpretation of complex numbers and the Argand diagram.

Example 1

The vectors **u** and **v** with corresponding points U and V are shown on an Argand diagram below.



Use the parallelogram rule to draw the vectors **w** and **z**, where

a $\mathbf{w} = \mathbf{u} + \mathbf{v}$ b $\mathbf{z} = \mathbf{u} - \mathbf{v}$

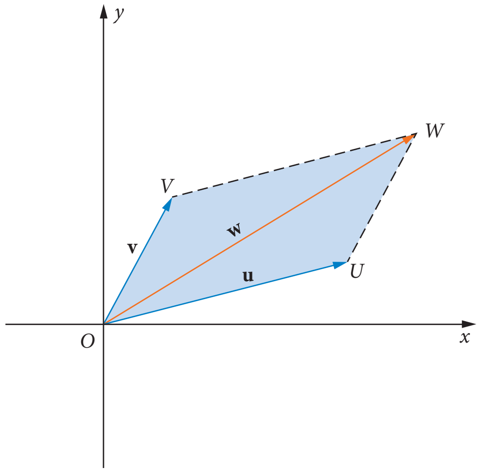
Plot the points W and Z that correspond to the vectors **w** and **z**.

9780170250306

Solution

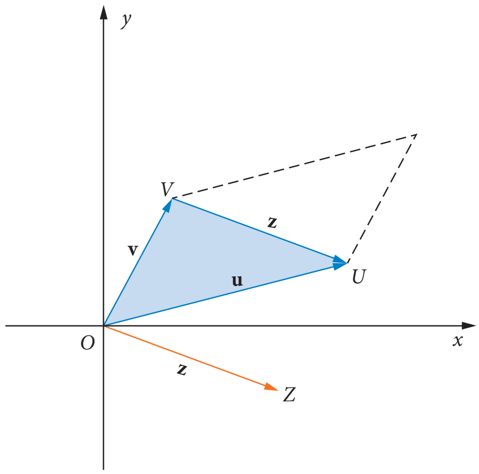
a To find the sum of two vectors, complete the parallelogram and draw the diagonal with its tail at O.

$\mathbf{w} = \mathbf{u} + \mathbf{v}$



b To find the difference of two vectors, $\mathbf{z} = \mathbf{u} - \mathbf{v}$, complete the parallelogram and draw the diagonal from V to U. Redraw $\mathbf{z}$ with its tail at the origin to find the point Z.

$\mathbf{z} = \mathbf{u} - \mathbf{v}$

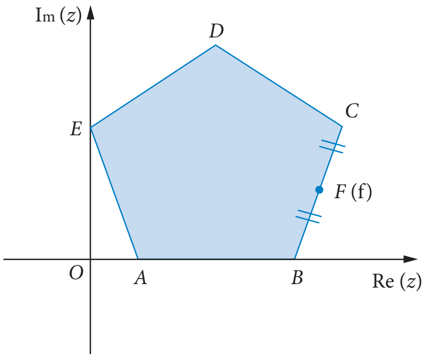


Example 2

Consider the vertices of the regular pentagon $ABCDE$ with position vectors **a**, **b**, **c**, **d**, **e** as shown. F (**f**) is the midpoint of BC.

Find expressions for each vector below in terms of **a**, **b**, **c**, **d** and **e.**

a $\overrightarrow{AB}$ **b** $\overrightarrow{AC}$ **c** $\overrightarrow{EB}$ **d** $\overrightarrow{DA}$ **e** $\overrightarrow{OF}$



Solution

a The vector $\overrightarrow{AB}$ has length AB, with its arrow at B.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$$

b $\overrightarrow{AC}$ is the sum of vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$. Alternatively, you can use the parallelogram rule.

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AB} + \overrightarrow{BC}\\ &= (\mathbf{b} - \mathbf{a}) + (\mathbf{c} - \mathbf{b})\\ &= \mathbf{c} - \mathbf{a}\end{aligned}$$

c Use the parallelogram rule.

$$\overrightarrow{EB} = \mathbf{b} - \mathbf{e}$$

d Use the parallelogram rule.

$$\overrightarrow{DA} = \mathbf{a} - \mathbf{d}$$

e Note that $\overrightarrow{BF} = \frac{1}{2}\overrightarrow{BC}$

$$\begin{aligned}\overrightarrow{OF} &= \overrightarrow{OB} + \tfrac{1}{2}\overrightarrow{BC}\\ &= \mathbf{b} + \tfrac{1}{2}(\mathbf{c} - \mathbf{b})\\ &= \frac{\mathbf{b} + \mathbf{c}}{2}\end{aligned}$$

Example 3

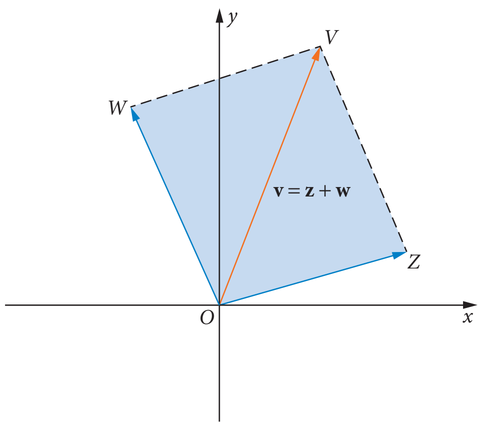
Consider the points Z and W representing the complex numbers $z = 3 + i$ and $w = -2 + 6i$ on the Argand plane. $OZVW$ is a parallelogram.

a Find the complex number v that represents the point V.

b Find the length of the diagonal ZW.

Solution

a OZ and OW are sides of $OZVW$. Use the parallelogram rule to find the complex number v that represents the point V.



Add the complex numbers.

$$\begin{aligned}v &= z + w\\ &= (3 + i) + (-2 + 6i)\\ &= 1 + 7i\end{aligned}$$

9780170250306

b Again, using the parallelogram rule, the length of the diagonal ZW is represented by the modulus of the vector $\mathbf{z} - \mathbf{w}$.

$$z - w = (3 + i) - (-2 + 6i)$$
$$= 5 - 5i$$

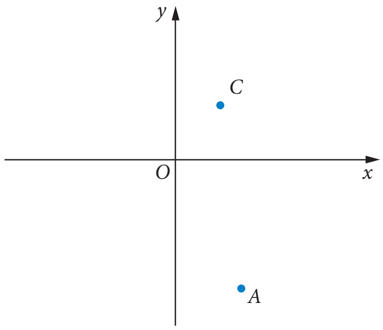
$$\therefore ZW = |5 - 5i|$$
$$= \sqrt{5^2 + (-5)^2}$$
$$= \sqrt{50}$$
$$= 5\sqrt{2}$$

EXERCISE 4.01 Addition and subtraction in the complex plane

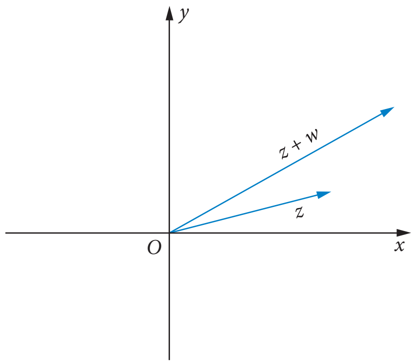
Concepts and techniques

WS
Addition and subtraction in the plane

1 Example 1 The points A, B, C correspond to the complex numbers a, b, c. $ABCO$ forms a parallelogram.
Copy the diagram. On your diagram,
   - **a** plot the point B
   - **b** express the vector $\mathbf{b}$ in terms of $\mathbf{a}$ and $\mathbf{c}$
   - **c** express the vector $\overrightarrow{AC}$ in terms of $\mathbf{a}$ and $\mathbf{c}$
   - **d** plot the point D that corresponds to the vector $\mathbf{c} - \mathbf{a}$.



2 The vectors $\mathbf{z}$ and $\mathbf{z} + \mathbf{w}$ corresponding to the complex numbers z and $z + w$ are plotted on an Argand diagram.
   - **a** Draw the vector $\mathbf{w}$ and plot the location of the point W representing w.
   - **b** With reference to the diagram, explain why $|z + w| \leq |z| + |w|$.



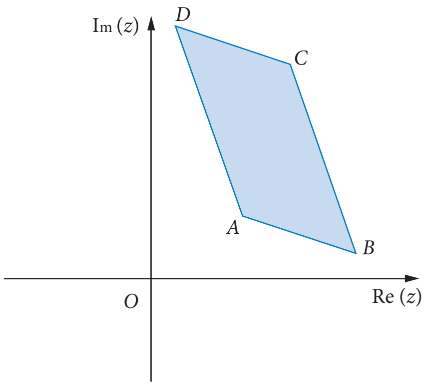
3 Use a similar technique to that in question **2** to show, for any two complex numbers z and w, that $|z - w| \geq |z| - |w|$. When does equality hold?

4 Example 2 Consider the parallelogram $ABCD$ representing the complex numbers z_1, z_2, z_3, z_4 respectively.

i Express each vector in terms of any two of z_1, z_2, z_3, z_4:

a $\overrightarrow{AB}$ b $\overrightarrow{CB}$ c $\overrightarrow{AC}$

ii Find the complex number w in terms of z_1, z_2, z_3 or z_4 that is represented by the midpoint W of DB.



5 In the diagram, $ABCD$ is a rhombus. The points A, B, C and D represent the complex numbers z, w, u and v.

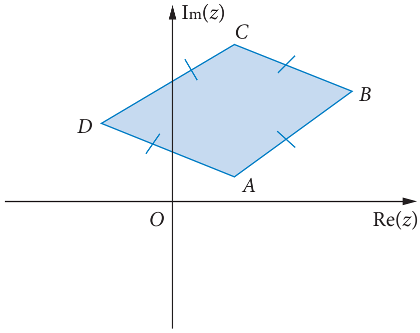
i Find, in terms of z, w, u or v,

a $\overrightarrow{AB}$ b $\overrightarrow{CB}$ c $\overrightarrow{AC}$

ii Find, in terms of z, w and v,

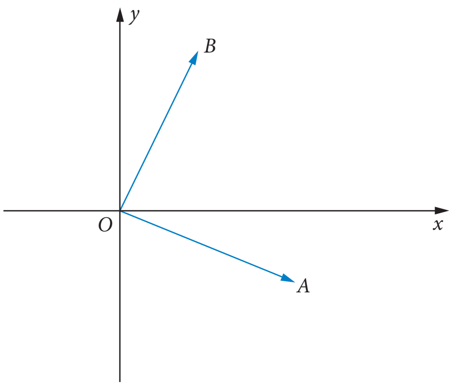
a u

b the complex number m represented by the midpoint M of AC.



6 P, Q, R, S representing the complex numbers w_1, w_2, w_3, w_4 form a quadrilateral in the complex plane. What type of quadrilateral is $PQRS$ if $w_1 - w_2 = w_4 - w_3$?

7 Example 3 Consider the complex numbers $a = \sqrt{3} - i$ and $b = 1 + i\sqrt{3}$ represented by the points A and B on an Argand diagram such that $OACB$ is a quadrilateral.



Copy the diagram and

a plot the point C such that $OACB$ is a parallelogram

b find the complex number c represented by C

c show that $|a| = |b|$

d show that $|a - b| = |c|$

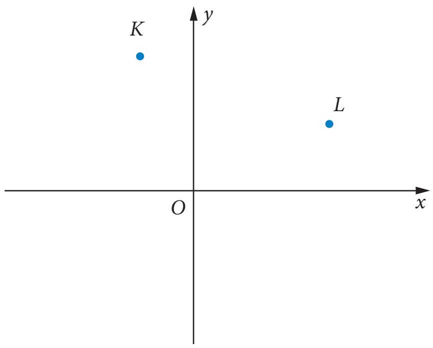
e classify the quadrilateral $OACB$.

9780170250306

Reasoning and communication

8 Plot the points A, B corresponding to the complex numbers $a = 4 - 6i$ and $b = 3 + 2i$.
   a Find $|a|$, $|b|$ and $|a - b|$.
   b Prove that $OA^2 + OB^2 = AB^2$.
   c What type of triangle is OAB?

9 The points L and K on the right represent the complex numbers z_1 and z_2.
Copy the diagram and plot the points M and N representing the complex numbers $-z_1$ and $-z_2$.
What type of quadrilateral is $LKMN$? Explain.



10 The complex numbers $w = 5[\cos(60°) + i\sin(60°)]$ and $z = 5[\cos(30°) + i\sin(30°)]$ are represented by points A and D on an Argand diagram. The quadrilateral $OACD$ is a rhombus.
   a Find the complex number c corresponding to the vertex C. Give your answer in Cartesian form.
   b Find the area of $OACD$ in simplest form.

4.02 MULTIPLICATION IN THE COMPLEX PLANE

Geometric interpretation in the plane

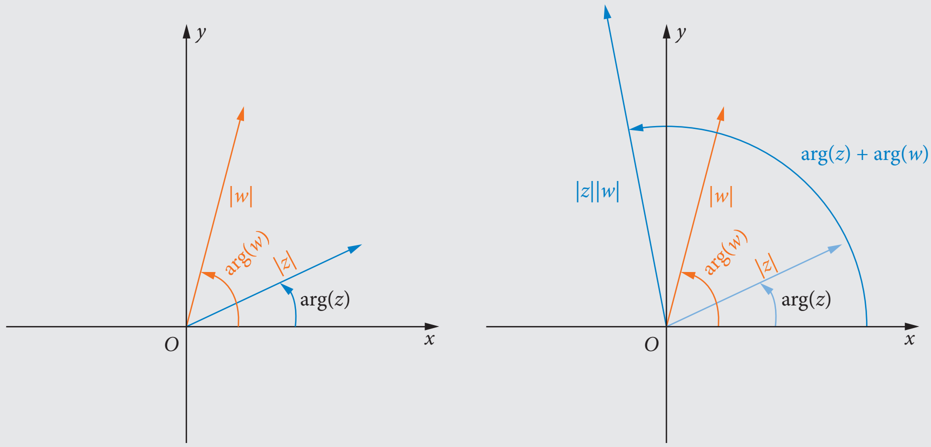
To multiply complex numbers, you multiply the moduli and add the arguments.

IMPORTANT

If $z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)]$ are two complex numbers, then their **product** is $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$.

Geometrically, the product of two vectors **z** and **w** have a length of $|z||w|$ and an argument equal to the sum $\arg(z) + \arg(w)$. If the vector **z** is multiplied by a vector **w**, its modulus increases by the factor $|w|$ and its argument is rotated by $\arg(w)$.

IMPORTANT



If $z = |z|\{\cos[\arg(z)] + i\sin[\arg(z)]\}$ and $w = |w|\{\cos[\arg(w)] + i\sin[\arg(w)]\}$ are two complex numbers, then their **product** is

$$zw = |z||w|\{\cos[\arg(z) + \arg(w)] + i\sin[\arg(z) + \arg(w)]\}$$

Example 4

If $z = \sqrt{2}\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$ and $w = i$, sketch the vector **zw** in the complex plane.

Solution

Convert $w = i$ to mod-arg form.

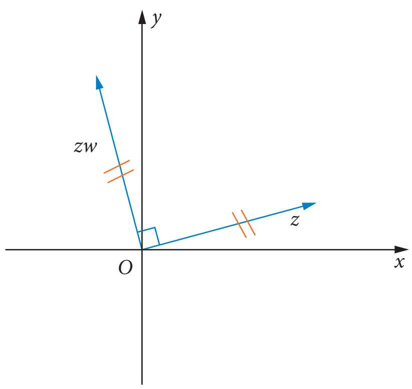
$\text{mod}(w) = 1$ and $\arg(w) = \frac{\pi}{2}$

$\therefore w = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$.

Use the rule $zw = |z||w|\{\cos[\arg(z) + \arg(w)] + i\sin[\arg(z) + \arg(w)]\}$

$$zw = \sqrt{2} \times 1\left[\cos\left(\frac{\pi}{6} + \frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{6} + \frac{\pi}{2}\right)\right]$$
$$= \sqrt{2}\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$$

z has been rotated $\frac{\pi}{2}$ but the length remains constant.

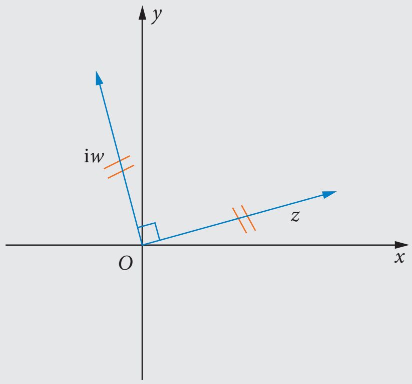


The example above is an illustration of the following rule.

9780170250306

IMPORTANT

Multiplication by i is equivalent to a rotation of $\frac{\pi}{2}$ in the complex plane.



By implication, multiplication by $-i$ is equivalent to a rotation of $-\frac{\pi}{2}$ in the complex plane.

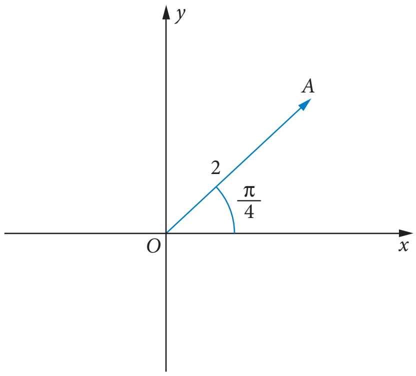
Example 5

If $z = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$ and $w = 3\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$, sketch the points A, B, C, D, E, corresponding to each complex number on the complex plane.

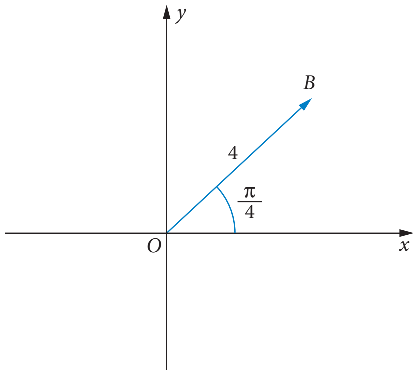
a $A = z$ **b** $B = 2z$ **c** $C = zw$ **d** $D = -z$ **e** $E = w^2$

Solution

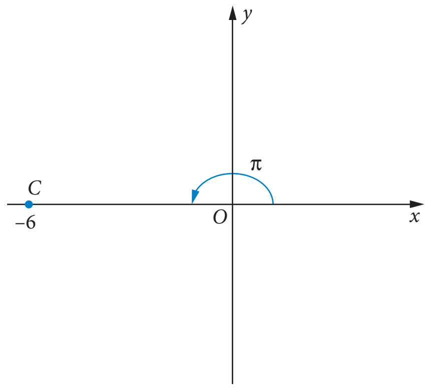
a Sketch a complex number with modulus 2 and argument $\frac{\pi}{4}$.



b The modulus of z is multiplied by a factor of 2.



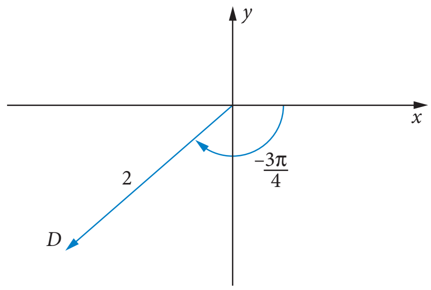
c Using the rule $zw = |z||w|\{\cos[\arg(z) + \arg(w)] + i\sin[\arg(z) + \arg(w)]\}$,
mod $(zw) = 2 \times 3 = 6$ and
$\arg(zw) = \frac{\pi}{4} + \frac{3\pi}{4} = \pi$



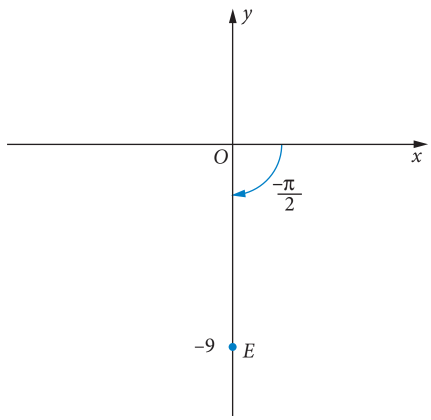
d $-z$ is equivalent to

$$-1 \times z = [\cos(\pi) + i\sin(\pi)] \times 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$$
$$= 2\left[\cos\left(\pi + \frac{\pi}{4}\right) + i\sin\left(\pi + \frac{\pi}{4}\right)\right]$$
$$= 2\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$
$$= 2\left[\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right]$$

Recall that the principle argument is in the interval $(-\pi, \pi]$.



e
$$w^2 = w \times w$$
$$= 3\,\text{cis}\left(\frac{3\pi}{4}\right) \times 3\,\text{cis}\left(\frac{3\pi}{4}\right)$$
$$= 9\,\text{cis}\left(\frac{3\pi}{2}\right)$$
$$= 9\,\text{cis}\left(\frac{-\pi}{2}\right)$$
$$= -9i$$



In Example **5**, you could use De Moivre's theorem to evaluate w^2. In that case,

$$w^2 = \left\{3\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]\right\}^2$$
$$= 3^2\left[\cos\left(2 \times \frac{3\pi}{4}\right) + i\sin\left(2 \times \frac{3\pi}{4}\right)\right]$$
$$= 9\left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right]$$
$$= 9\left[\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right]$$
$$= -9$$

Example 6

Given $\omega = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$, sketch ω, ω^2 and ω^3 on the same axes.

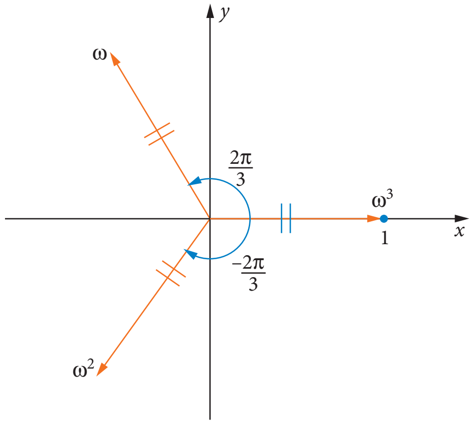
Solution

$\arg(\omega) = \frac{2\pi}{3}$ and $|\omega| = 1$.

Using De Moivre's theorem,

$$\begin{aligned}\omega^2 &= \left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]^2\\ &= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\\ &= \cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\end{aligned}$$

$$\begin{aligned}\omega^3 &= \left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]^3\\ &= \cos\left(\frac{6\pi}{3}\right) + i\sin\left(\frac{6\pi}{3}\right)\\ &= \cos(2\pi) + i\sin(2\pi)\\ &= 1\end{aligned}$$



EXERCISE 4.02 Multiplication in the complex plane

Concepts and techniques

WS Multiplication in the plane

1 Example 4 Find the resulting vector when each of the following is multiplied by the complex number i.

a $\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$

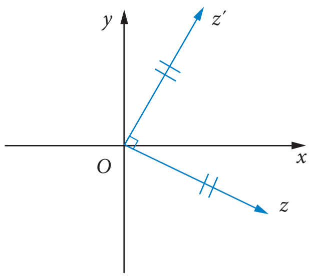
b $\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$

c $2\left[\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right]$

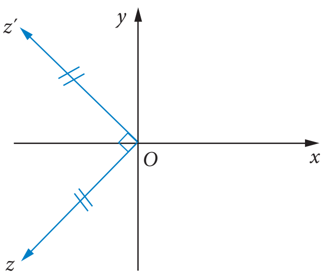
d $\sqrt{3}\left[\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right]$

2 In each of the following, multiplication by which complex number takes z to z'?

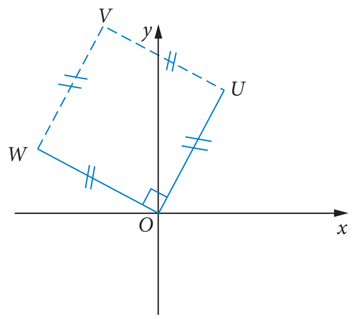
a



b



3 In the diagram, $UVWO$ is a square. The point U represents the complex number u.

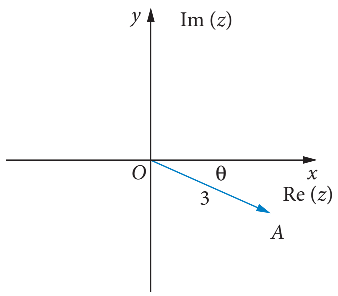


Find, in terms of u, the complex numbers represented by

a W

b V

4 Example 5 The point A represents the complex number $z = 3 \text{ cis } (-\theta)$ on an Argand diagram.



Sketch on the same diagram:

a $-z$

b z^2

c iz

5 The complex numbers $z = \sqrt{2}\left[\cos(30°) + i\sin(30°)\right]$ and $w = \sqrt{3}\left[\cos(120°) + i\sin(120°)\right]$ are represented by the points Z and W on the complex plane. Sketch the point U representing the product zw.

6 Consider the parallelogram $OABC$, where A, B, C represent the complex numbers a, b, c. If $a = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$, $OC = 2OA$ and $\angle COA = \frac{\pi}{3}$, find the complex numbers c and b.

7 Example 6 Given that $z = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$, sketch the following on an Argand diagram.

a z

b z^2

c z^3

d z^4

e z^5

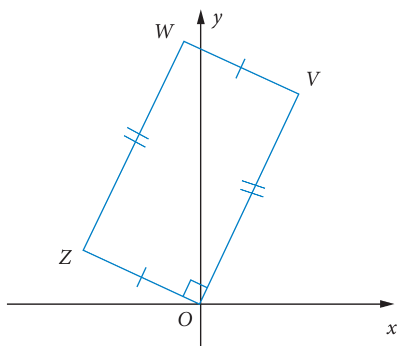
f z^6

Reasoning and communication

8 Consider the rectangle $OZWV$. Given that Z represents the complex number z and $OV = 2OZ$,

a find the complex numbers represented by the points V and W in terms of z

b find the complex number in terms of z represented by the point U, the intersection of OW and ZV.



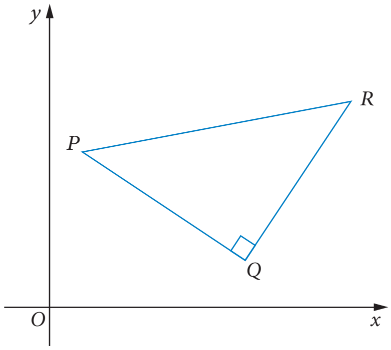
9780170250306

9 The complex numbers w_1, w_2 and w_3 correspond to the vertices A, B and C respectively of an arbitrary triangle.

a Draw a diagram and label the vectors $w_1 - w_2$ and $w_3 - w_2$.

b If $w_1 - w_2 = i(w_3 - w_2)$, what type of triangle is $\triangle ABC$?

10 Consider the right-angled isosceles triangle PQR with corresponding complex numbers z_1, z_2, z_3 respectively. The complex number z_4 represented by the point S is defined by $z_4 = z_1 + i(z_2 - z_1)$.



a Plot the point S.

b Show that $z_4 - z_3 = z_1 - z_2$.

c What shape is $SPQR$?

4.03 DIVISION IN THE COMPLEX PLANE

Geometric interpretation in the plane

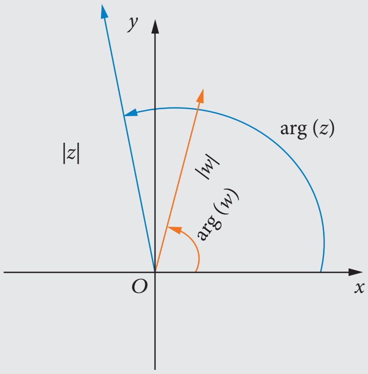
To divide complex numbers, you divide the moduli and subtract the arguments.

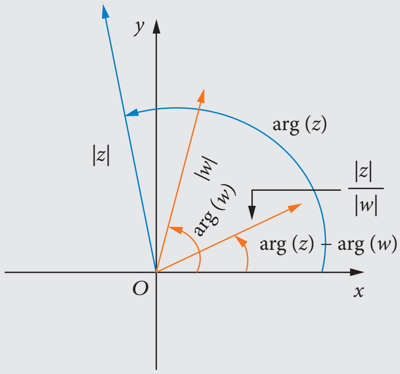
IMPORTANT

If $z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)]$ are two complex numbers, then their **quotient** is $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$.

Geometrically, this means that the quotient of two numbers z and w have a length of $\frac{|z|}{|w|}$ and an argument equal to the difference $\arg(z) - \arg(w)$. Alternatively, if a number z is divided by a number w, then its modulus is decreased by the factor $|w|$ and its argument is rotated through the angle $\arg(w)$. If $\arg(z)$ is *positive*, the direction is *anticlockwise*. If it is negative, then the rotation will be clockwise.

IMPORTANT





If $z = |z|\{\cos[\arg(z)] + i\sin[\arg(z)]\}$ and $w = |w|\{\cos[\arg(w)] + i\sin[\arg(w)]\}$ are two complex numbers, then their **quotient** is

$$\frac{z}{w} = \frac{|z|}{|w|}\{\cos[\arg(z) - \arg(w)] + i\sin[\arg(z) - \arg(w)]\}$$

Example 7

If $z = 4[\cos(120°) + i\sin(120°)]$ and $w = 2[\cos(30°) + i\sin(30°)]$, sketch z and w, and show that $\frac{z}{w}$ is purely imaginary.

Solution

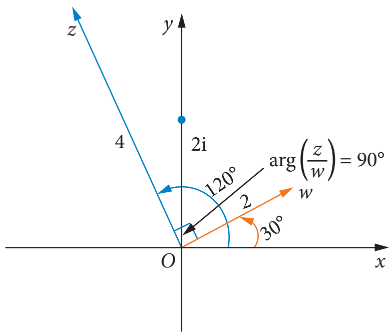
$|z| = 4$, $|w| = 2$

$\arg(z) = 120°$, $\arg(w) = 30°$

$$\text{mod}\left(\frac{z}{w}\right) = \frac{|z|}{|w|} = \frac{4}{2} = 2$$

$$\arg\left(\frac{z}{w}\right) = 120° - 30° = 90°$$

$\frac{z}{w} = 2[\cos(90°) + i\sin(90°)] = 2i$, which is purely imaginary, i.e., $2i$ lies on the y-axis.



9780170250306

Example 8

If $z = r\text{ cis }(\theta)$, sketch

a $-iz$ **b** $\frac{z}{i}$

Solution

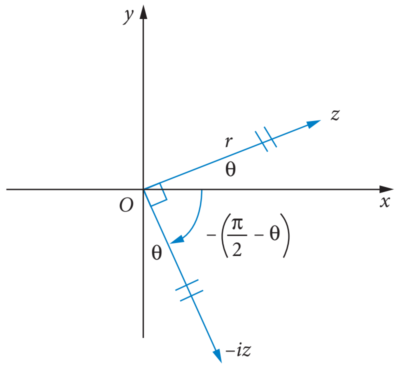
a First express $-i$ in polar form.

$$-i = \cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)$$

Use the rule $zw = |z||w|\{\cos[\arg(z) + \arg(w)] + i\sin[\arg(z) + \arg(w)]\}$

$$-iz = \text{cis}\left(\frac{-\pi}{2}\right) \times r\,\text{cis}(\theta)$$
$$= r\,\text{cis}\left(\frac{-\pi}{2} + \theta\right)$$
$$= r\,\text{cis}\left[-\left(\frac{\pi}{2} - \theta\right)\right]$$

$-iz$ is equivalent to a rotation of z through an angle of $\frac{-\pi}{2}$.

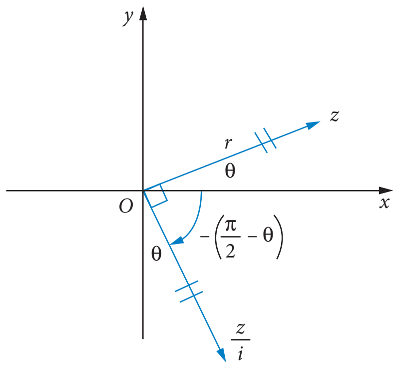


b Express $\frac{z}{i}$ as a quotient of two complex numbers using the rule

$$\frac{z}{w} = \frac{|z|}{|w|}\{\cos[\arg(z) - \arg(w)] + i\sin[\arg(z) - \arg(w)]\}$$

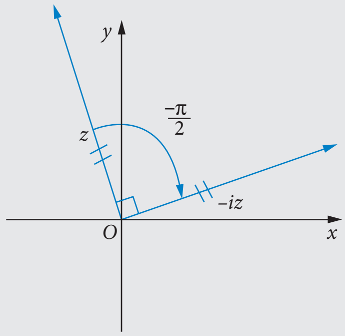
$$\frac{z}{i} = \frac{r\,\text{cis}(\theta)}{\text{cis}\left(\frac{\pi}{2}\right)}$$
$$= r\,\text{cis}\left(\theta - \frac{\pi}{2}\right)$$
$$= r\,\text{cis}\left[-\left(\frac{\pi}{2} - \theta\right)\right]$$
$$= -iz$$

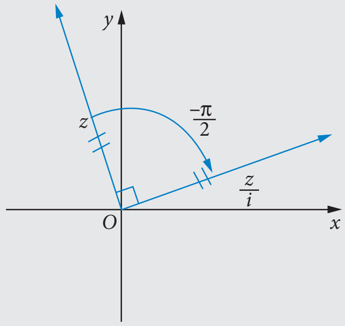
Note that both $-iz$ and $\frac{z}{i}$ rotate z through an angle of $\frac{-\pi}{2}$.



IMPORTANT

Multiplication by $-i$ or division by i are transformations equivalent to a rotation of $\frac{-\pi}{2}$ in the complex plane.

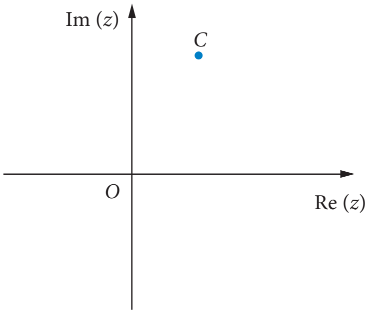




It is not difficult to see why this result is true: $\frac{z}{i}=\frac{z}{i}\times\frac{i}{i}=\frac{iz}{-1}=-iz$.

Example 9

If the point C represents the complex number c, plot the point R that represents $\frac{1}{c}$.

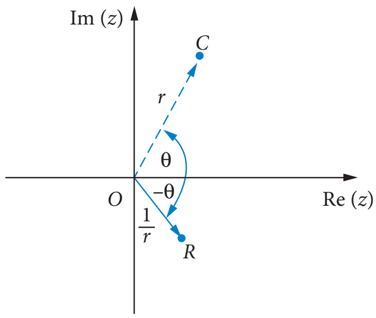


Solution

First express the complex number c as $c = r\text{ cis }(\theta)$.

$$\frac{1}{c}=c^{-1}$$
$$=\left[r\text{cis}(\theta)\right]^{-1}$$
$$=r^{-1}\text{cis}(-\theta)$$
$$=\frac{1}{r}\text{ cis}(-\theta)$$

The modulus of $\frac{1}{c}$ is $\frac{1}{r}$ and the argument is $-\theta$.



9780170250306

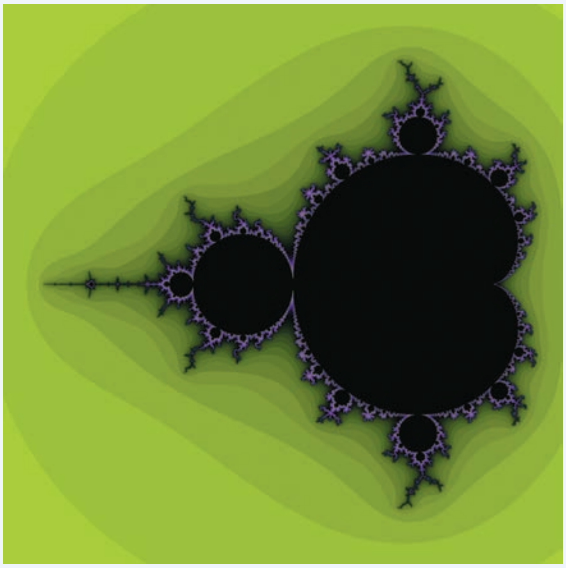
INVESTIGATION The development of chaos theory

It was not until the 1970s, with the advent of powerful computers that could do a multitude of calculations in a split second, that chaos theory was proposed. A mathematician, Benoit Mandelbrot, working for the large computer firm IBM, noticed that repeating a formula similar to Pythagoras' theorem using complex numbers generated images similar to those that occur in nature. He found that a simple equation could model ferns and trees, clouds and landforms, river systems and weather patterns. He began to wonder whether there is a pattern to observations that appear to be chaotic. Thus chaos theory was born.

Try to find out about each of the following.

- The Von Koch Curve
- The Mandelbrot set
- Fractals and their use.

There are some interesting properties of fractals. Find out what they are.



Shutterstock.com/Markus Gann

Generating the Mandelbrot set

EXERCISE 4.03 Division in the complex plane

Concepts and techniques

Division in the plane

1 Example 7 For each pair of complex numbers z_1 and z_2, find $\left|\frac{z_1}{z_2}\right|$ and $\arg\left(\frac{z_1}{z_2}\right)$. Hence sketch $\frac{z_1}{z_2}$ on an Argand diagram.

a $z_1 = 2\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$ and $z_2 = \sqrt{2}\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$

b $z_1 = 5\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$ and $z_2 = \left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$

c $z_1 = 9\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]$ and $z_2 = 3\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$

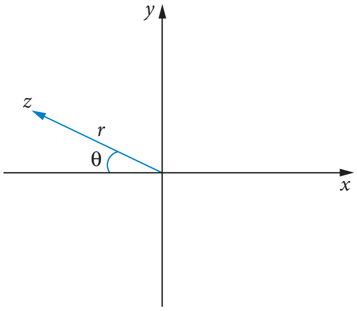
d $z_1 = 4\left[\cos\left(\frac{-5\pi}{12}\right) + i\sin\left(\frac{-5\pi}{12}\right)\right]$ and $z_2 = 8\left[\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right]$

2 Find $\frac{z_1}{z_2}$ if $z_1 = 8\left[\cos\left(\frac{3\pi}{4}\right) - i\sin\left(\frac{3\pi}{4}\right)\right]$ and $z_2 = 2\sqrt{2}\left[\cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)\right]$. Sketch $\frac{z_1}{z_2}$.

3 By first expressing it in polar form, sketch $\dfrac{z_1 z_2}{z_3}$ on the Argand plane if

$z_1 = 4\left[\cos\left(\dfrac{-\pi}{3}\right) + i\sin\left(\dfrac{-\pi}{3}\right)\right]$, $z_2 = 3\left[\cos\left(\dfrac{\pi}{6}\right) - i\sin\left(\dfrac{\pi}{6}\right)\right]$ and $z_3 = 6\left[\cos\left(\dfrac{2\pi}{3}\right) + i\sin\left(\dfrac{2\pi}{3}\right)\right]$.

4 Example 8 Given the vector $\mathbf{z}$ representing the complex number z on the complex plane, sketch each of the following in relation to z.



a iz

b $\dfrac{z}{i}$

c $-iz$

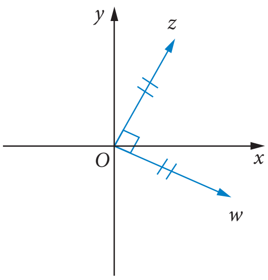
d $\dfrac{z}{i^2}$

e $-z$

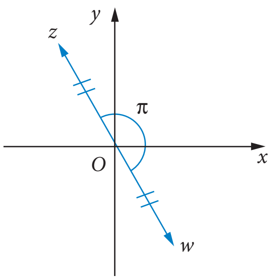
In each case, state which transformations of z are equivalent.

5 In each case, express the complex number w in terms of z.

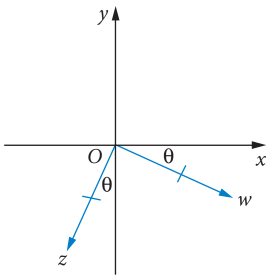
a



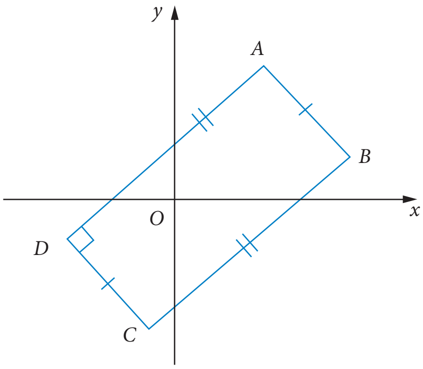
b



c



6 Consider the complex numbers w_1, w_2, w_3, w_4, which correspond to the vertices of a rectangle $ABCD$, where $AD = 2AB$. Explain why $\dfrac{w_2 - w_1}{i} = \dfrac{w_4 - w_1}{2}$.



7 Example 9 Given each complex number z, sketch $\dfrac{1}{z}$.

a $z = 4\,\text{cis}\left(\dfrac{\pi}{3}\right)$

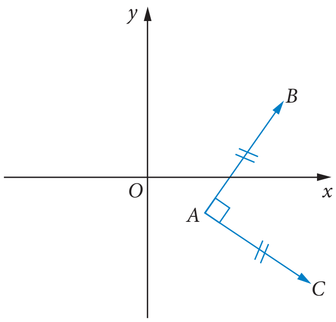
b $z = \sqrt{2}\,\text{cis}\left(\dfrac{5\pi}{6}\right)$

c $z = \dfrac{1}{\sqrt{3}}\,\text{cis}\left(\dfrac{-\pi}{4}\right)$

9780170250306

Reasoning and communication

8 Consider the points A, B, C representing the complex numbers z_1, z_2, z_3 on the complex plane. Prove that $\dfrac{z_2 - z_1}{z_3 - z_1} = i$.



9 The arbitrary points L and M representing complex numbers w_2, w_3 together with O form a triangle in the complex plane.

a Plot the information and draw the point N representing $w_2 + w_3$.

b Describe the quadrilateral $OLNM$.

c Given that $|w_2 + w_3| = |w_2 - w_3|$, what can you deduce about $\dfrac{w_2}{w_3}$?

10 Prove that the complex numbers z_1, z_2 and $\dfrac{z_1 - iz_2}{1-i}$ form a right-angled triangle.

4.04 COMPLEX NUMBER OPERATIONS AS TRANSFORMATIONS

Vector addition and subtraction can be used to translate plane shapes, whereas multiplication and division can be used to rotate plane shapes.

IMPORTANT

For two non-zero complex numbers z_1 and z_2,

$z_1 = z_2$ *if and only if* $|z_1| = |z_2|$ AND $\arg(z_1) = \arg(z_2)$

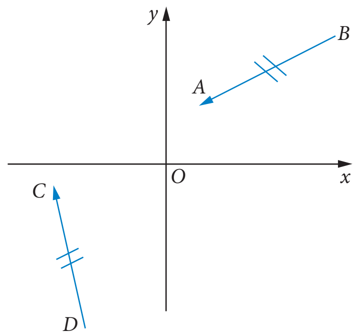
Example 10

Consider the complex numbers z_1, z_2, z_3, z_4 represented by the points A, B, C, D. What can you deduce from the following? Draw a diagram.

a $|z_1 - z_2| = |z_3 - z_4|$ **b** $\arg(z_1 - z_2) = \arg(z_3 - z_4)$ **c** $z_1 - z_2 = z_3 - z_4$

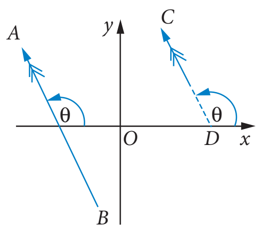
Solution

a $|z_1 - z_2| = |z_3 - z_4|$ means that the lengths BA and DC are equal.



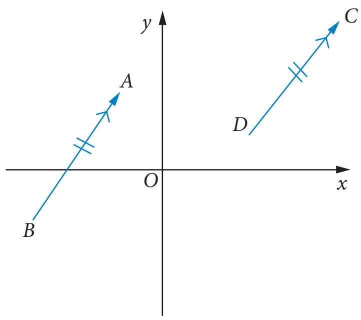
$AB = CD$.

b $\arg(z_1 - z_2) = \arg(z_3 - z_4)$ means that the vectors $z_1 - z_2$ and $z_3 - z_4$ are running in the same direction.

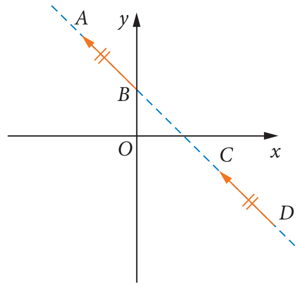


$AB \parallel CD$

c $z_1 - z_2 = z_3 - z_4$ means that the vectors $z_1 - z_2$ and $z_3 - z_4$ are the same length and have the same argument.

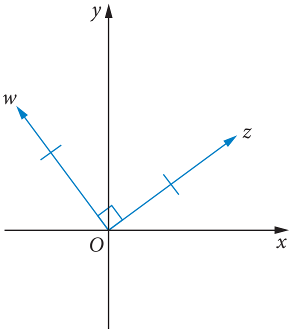


Either $ABDC$ forms a parallelogram or A, B, C, D are collinear and $AB = CD$.



Example 11

Consider the complex numbers with corresponding vectors sketched below.



Prove that $z^2 + w^2 = 0$.

Solution

Since the angle between them is 90°, $w = iz$.

$w = iz$.

Squaring both sides,

$$w^2 = (iz)^2 = i^2z^2 = -z^2$$

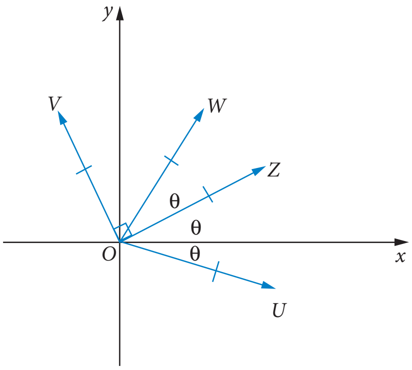
Rearrange to get the desired result.

$\therefore z^2 + w^2 = 0$.

Example 12

Consider the diagram with complex numbers z, w, u, v represented by points Z, W, U, V respectively.

Express w, u, v in terms of z.



Solution

From the diagram,

$|z| = |w| = |u| = |v|$.

Also, $\arg(z) = \theta$, whereas $\arg(w) = 2\theta$.

$$|z| = |w|$$

$$\arg(w) = \arg(z) + \arg(z)$$

$$\therefore w = |z| \operatorname{cis}(\theta + \theta) = |z| \operatorname{cis}(\theta) \operatorname{cis}(\theta) = z \operatorname{cis}(\theta) = z \operatorname{cis}[\arg(z)]$$

U is the reflection of Z over the real axis. $u = \overline{z}$

V is the result of rotating Z through 90°. $v = iz$

EXERCISE 4.04 Complex number operations as transformations

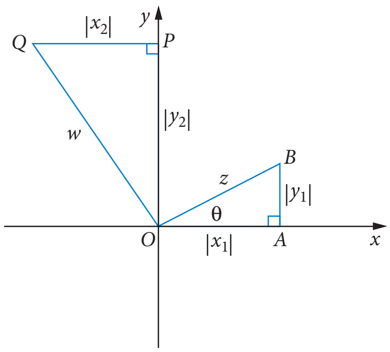
Concepts and techniques

1 Example 10 Draw sketches to illustrate the following relationships for complex numbers. In each case, give a description.

a $\arg(z) = \arg(u)$ b $|z| = |w|$ c $\arg(z) = -\arg(u)$ d $\arg(z - w) = \arg(u - v)$
e $|z - u| = |w - v|$ f $z - w = u - v$ g $\arg(z) - \arg(u) = 0$ h $\arg(z) + \arg(u) = 0$
i $|z + u| = |z - u|$ j $z + w = u + v$ k $z - w = i(z + w)$

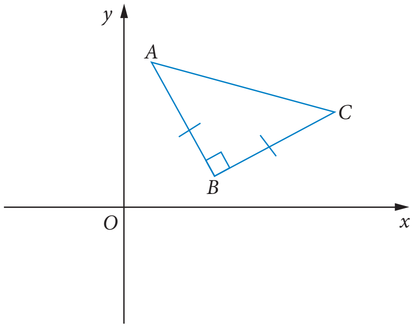
2 If, for a complex number z, $\arg(z) = \theta$, find $\arg(-z)$ in terms of θ.

3 Consider the similar triangles OAB and OPQ in the diagram on the right.
B and Q represent the complex numbers z and w respectively. Let $\arg(z) = \theta$.
Show that $w|y_1| = iz|x_2|$.

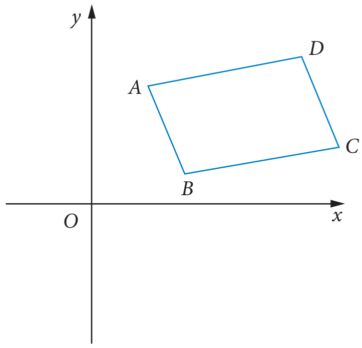


4 Example 11 In the Argand diagram, ABC is a right-angled isosceles triangle.
If A, B, C represent the complex numbers w_1, w_2, w_3 respectively, prove that

a $|w_1 - w_2| = |w_3 - w_2|$
b $\arg(w_1 - w_2) - \arg(w_3 - w_2) = \dfrac{\pi}{2}$
c $(w_1 - w_2)^2 + (w_3 - w_2)^2 = 0$
d $|w_1 - w_2|^2 + |w_3 - w_2|^2 = |w_3 - w_1|^2$



5 Consider the parallelogram $ABCD$ representing the complex numbers z_1, z_2, z_3, z_4 respectively.
Plot the point:
a E corresponding to the complex number $z_3 - z_4$
b F corresponding to the complex number $z_1 - z_3$
c G corresponding to the complex number $-i(z_2 - z_1)$



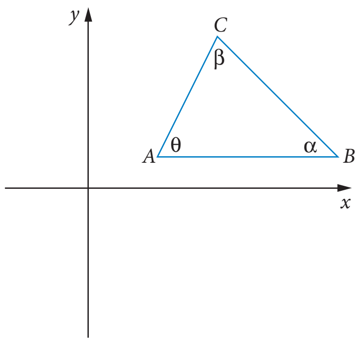
9780170250306

6 Consider the triangle ABC whose vertices represent z_1, z_2, z_3 respectively. AB is parallel to the x-axis. Show that

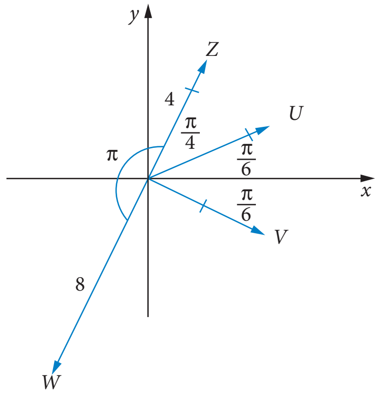
a $\arg(z_3 - z_1) = \theta$

b $\arg(z_3 - z_2) = \pi - \alpha$

c $\arg\left(\dfrac{z_3 - z_2}{z_3 - z_1}\right) = \beta$



7 Example 12 Consider the complex numbers z, w, u, v represented by the points Z, W, U, V in the complex plane.
Express z, w, v in terms of u.



Reasoning and communication

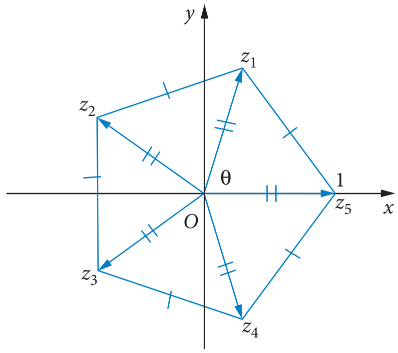
8 Consider the complex numbers u, v and w such that $|u| = |v| = 1$, $\arg(u) = \theta$ and $\arg(v) = \alpha$. Find an expression for w in terms of u and v if

a $\arg(w) = \theta - \alpha$

b $\arg(w) = \theta + \alpha$

Draw a diagram in each case.

9 Consider the complex numbers sketched below.



a Find the value of θ.

b Write down the complex numbers z_1, z_2, z_3, z_4, z_5.

c Show that $z_1^2 = z_2$

d Show that $z_2^2 = z_4$

e Show that $z_3^2 = z_1$

f Show that $\overline{z_1} = z_4$

4.05 GRAPHING SUBSETS OF THE COMPLEX PLANE: SUBSETS INVOLVING MODULUS

Subsets of the complex plane are sets of points representing complex numbers. These are described by a certain rule or condition placed on the variable complex number z. You can use a geometric approach or an algebraic approach to answering these questions. The set of points is often called the **locus of** z.

Example 13

Sketch the locus of z if

a $|z + 2i| = 4$ **b** $|z - 3| = 1$ **c** $1 \le |z| < 9$

Solution

a *Geometric approach*

Write $|z + 2i| = |z - (0 - 2i)|$, which means the distance of z from the point $(0, -2)$.

If $|z - (0 - 2i)| = 4$, then z is the set of points 4 units from $(0, -2)$, or the circle with radius 4 and centre $(0, -2)$.

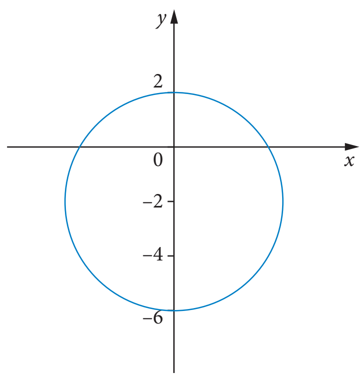
Algebraic approach

Let $z = (x, y)$, then $|z + 2i| = 4$ becomes $|x + yi + 2i| = 4$. Regrouping and using the definition of modulus,

$$|x + (y + 2)i| = 4$$

$$\sqrt{x^2 + (y+2)^2} = 4$$

$$x^2 + (y + 2)^2 = 16$$



The locus of z is $x^2 + (y + 2)^2 = 16$.

b *Geometric approach*

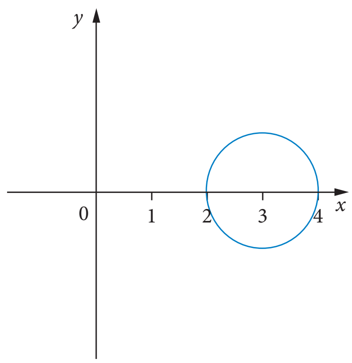
$|z - 3| = 1$ means that the distance of z from the point $(3, 0)$ is 1 unit.

The set of points on a circle of centre $(3, 0)$ and radius 1.

Algebraic approach

$|z - 3| = 1$ becomes $|(x - 3) + yi| = 1$.

So $(x - 3)^2 + y^2 = 1$ is the circle with radius 1 and centre $(3, 0)$.



The locus of z is $(x - 3)^2 + y^2 = 1$.

9780170250306

c *Geometric approach*

$1 \le |z| < 9$ means the distance of z from the origin is at least 1, but less than 9 units.
The set of points between the circles with centre (0, 0) and radii 1 and 9, including the circle of radius 1, but not that of radius 9.

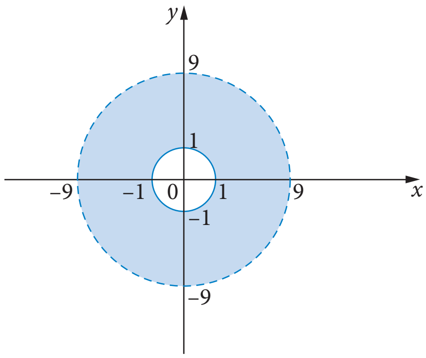
Algebraic approach

$1 \le |z| < 9$ becomes

$1 \le \sqrt{x^2 + y^2} < 9$

$1 \le x^2 + y^2 < 81$

So $1 \le x^2 + y^2 < 81$ is the region between the two circles with centre (0, 0) and radii 1 and 9, including the circle of radius 1, but not that of radius 9.



The locus of z is the area between the two concentric circles. The inequality can be written as $1 \le x^2 + y^2 < 81$.
Note the dotted circle since $|z| \ne 9$.

Some cases are more suited to the algebraic approach and other cases are better solved using the geometric approach.

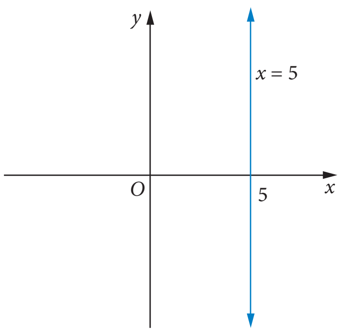
Example 14

Sketch z if

a Re $(z) = 5$ **b** Re $(z) =$ Im (z) **c** $|z - 2 - i| =$ Re (z)

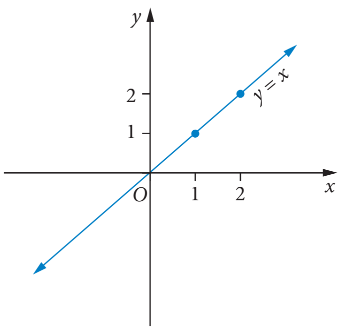
Solution

a If $z = x + yi$, then Re $(z) = x$, so if Re $(z) = 5$, then the locus is the vertical line $x = 5$.



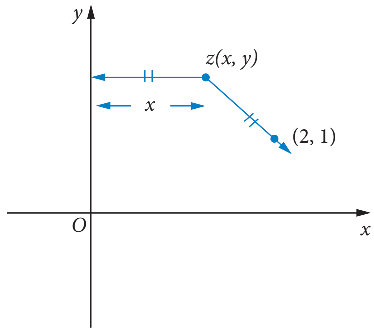
$x = 5$.

b If Re $(z) =$ Im (z), replace with $x = y$.
Therefore the locus is the line $y = x$.



$y = x$.

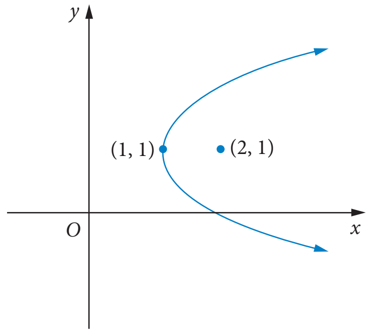
c *Geometrically*, $|z-2-i| = \text{Re}(z)$ means that the distance of z from the point (2, 1) is the same as the distance of z from the y-axis.
Using the locus definition of a parabola, (2, 1) is the focus and the y-axis is the directrix.
This means that (1, 1) is the vertex.



Algebraically, $\sqrt{(x-2)^2+(y-1)^2} = x$.
Squaring both sides and simplifying,

$$(x-2)^2 + (y-1)^2 = x^2$$
$$x^2 - 4x + 4 + (y-1)^2 = x^2$$
$$(y-1)^2 = 4x - 4$$

So $(y-1)^2 = 4(x-1)$, which is a parabola with vertex (1, 1) and focal length 1.



The locus of z is $(y-1)^2 = 4(x-1)$.

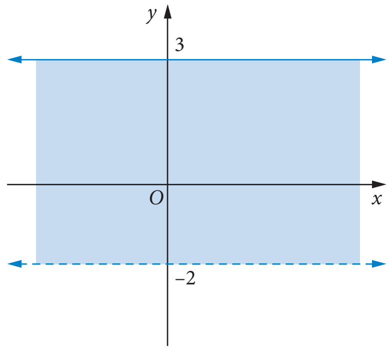
Example 15

Sketch z if

a $-2 < \text{Im}(z) \leq 3$ b $|z-2| = |z|$ c $|z| \geq 2$ and $-\frac{\pi}{3} \leq \arg(z) < \frac{\pi}{3}$

Solution

a Im (z) represents the y-value on the complex plane.
$-2 < \text{Im}(z) \leq 3$ can be written as $-2 < y \leq 3$, which represents a region on the complex plane.
Note that the line $y = -2$ is dotted as it is not included in the inequality.



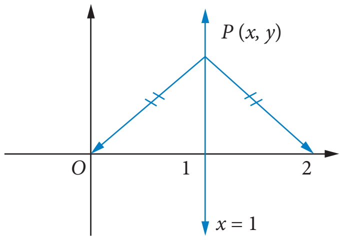
$-2 < \text{Im}(z) \leq 3$

9780170250306

b $|z-2| = |z|$ can be interpreted geometrically as the set of points equidistant from the point (2, 0) and the point (0, 0), or the perpendicular bisector of the interval joining those points. The equation is $x = 1$.

Algebraically, let $z = x + yi$ and

$$|x-2+yi| = |x+yi|$$
$$(x-2)^2 + y^2 = x^2 + y^2$$
$$4x - 4 = 0$$
$$x = 1$$

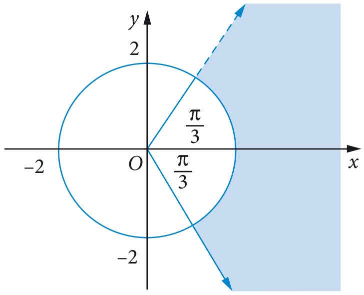


z is the set of points on the vertical line $x = 1$.

c $|z| \geq 2$ and $-\frac{\pi}{3} \leq \arg(z) < \frac{\pi}{3}$ is the intersection of points satisfying $|z| \geq 2$ and $-\frac{\pi}{3} \leq \arg(z) < \frac{\pi}{3}$.

$|z| \geq 2$ is the set of points on or outside the circle $|z| = 2$.

$-\frac{\pi}{3} \leq \arg(z) < \frac{\pi}{3}$ is the set of points with arguments from $-\frac{\pi}{3}$ to $\frac{\pi}{3}$, not including $\frac{\pi}{3}$.



$|z| \geq 2$ and $-\frac{\pi}{3} \leq \arg(z) < \frac{\pi}{3}$.

EXERCISE 4.05 Graphing subsets of the complex plane: subsets involving modulus

Concepts and techniques

1 Example 13 Sketch the circle defined by each equation below.
   a $|z| = 2$ b $|z| = 6$ c $|z| = 2\sqrt{2}$ d $|z| = \frac{1}{4}$
   e $|z-1| = 4$ f $|z+2| = 1$ g $|z-3i| = 3$ h $|z+i| = \frac{1}{2}$

2 Sketch each of the following.
   a $|z-(1+i)| = 1$ b $|z-(2-3i)| = 2$ c $|z-1-2i| = \frac{1}{2}$ d $|z+3-4i| = 2$

3 Sketch each region defined by the inequalities below.
   a $|z| \leq 3$ b $|z| > 1$ c $\frac{1}{4} < |z| \leq 4$ d $1 \leq |z-1| < 3$

4 Example 14 Sketch z for each of the following.
   a $\text{Re}(z) = 2$ b $\text{Im}(z) = 3$ c $\text{Re}(z) = -1$ d $\text{Im}(z) = -2$

5 Using the fact that $z = x + yi$, where x and y are real, find the Cartesian equation of each locus below and hence sketch
   a $\text{Re}(z) = -\text{Im}(z)$ b $\text{Im}(z) = 2\,\text{Re}(z)$ c $\text{Re}(z) = 3\,\text{Im}(z) - 1$ d $\text{Im}(z) + 4\,\text{Re}(z) = 12$

6 Sketch the parabola defined by each equation below.
   a $\text{Re}(z) = |z-2|$ b $\text{Im}(z) = |z-4i|$ c $|z-2-2i| = \text{Re}(z)$ d $|z-4+6i| = \text{Im}(z)$

7 Example 15 Sketch z defined by each inequality below.
   a $\text{Re}(z) > 4$ b $\text{Im}(z) < 3$ c $-1 < \text{Re}(z) \leq 1$ d $\text{Im}(z) < -2$ and $\text{Re}(z) \leq 4$

8 Sketch each inequality below.

a $|\text{Re}(z)| \le 1$ b $|\text{Im}(z)| < 3$ c $|\text{Re}(z)| \ge \frac{1}{2}$ d $|\text{Re}(z)| > 2$ and $|\text{Im}(z)| \le 1$

9 Describe each of the loci below and hence sketch:

a $|z| = |z - 2i|$ b $|z - 3| = |z + 3|$ c $|z - 1 - i| = |z + 1 + i|$ d $|z - 3 + i| = |z + 5 - 3i|$

10 Sketch each of the following regions.

a $|z| \le 2$ and $0 < \arg(z) < \frac{\pi}{2}$ b $|z - 1| \le 1$ and $0 < \arg(z) < \frac{\pi}{4}$

c $|z - 2i| \ge 1$ and $\frac{\pi}{3} < \arg(z) < \frac{2\pi}{3}$ d $1 \le |z| < 4$ and $-\frac{\pi}{4} < \arg(z) \le \frac{\pi}{4}$

Reasoning and communication

11 Sketch the following.

a $\left|\frac{z+2}{z-4}\right| = 1$ b $\left|\frac{z+i}{z-i}\right| \le 1$ c $\left|\frac{z+3}{2z}\right| > 1$

12 Show that the equation $|z - 3| = 2|z + 1|$ describes a circle. Find its centre and radius.

13 Sketch $|z - 3| + |z + 3| = 8$.

14 Find the maximum value of $\arg(z + 1)$ if $|z - 1| = 1$. Hint: draw and label a diagram.

4.06 GRAPHING SUBSETS OF THE COMPLEX PLANE: SUBSETS NOT INVOLVING MODULUS

Subsets not involving the modulus often involve the argument instead. Many of these sets are best approached geometrically. The beginning of a ray is not included as the argument for $|z| = 0$ is not defined.

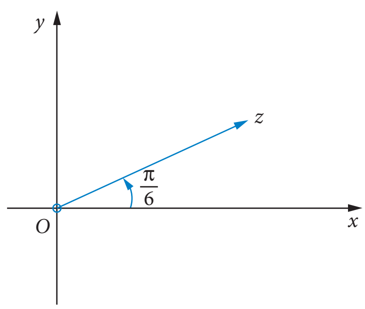
Example 16

Let z be a variable complex number in the complex plane. Sketch the locus of z if

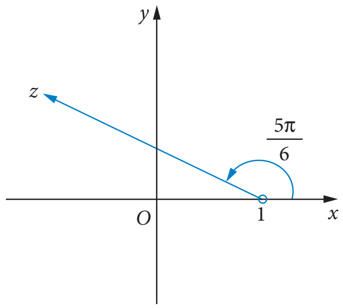
a $\arg(z) = \frac{\pi}{6}$ b $\arg(z - 1) = \frac{5\pi}{6}$ c $\arg[z - (2 + 3i)] = \frac{-\pi}{4}$

Solution

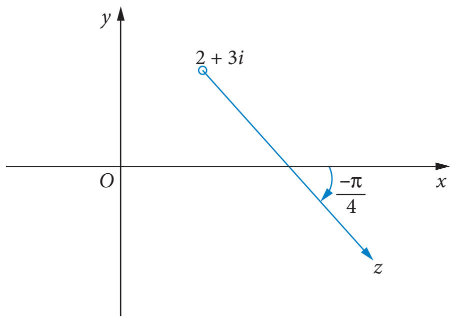
a Arg (z) is defined as the angle between the positive x-axis and the vector z, with its tail at the origin. In this case, z can be anywhere along the ray inclined at an angle of $\frac{\pi}{6}$ (going anticlockwise). Note: $z \ne 0$ as the argument of $z = 0$ is not defined.



b $(z-1)$ or $z-(1+0i)$ is the vector with its tail at $(1, 0)$. z can be anywhere along the ray inclined at $\frac{5\pi}{6}$.



c $z-(2+3i)$ is the vector with its tail at $(2+3i)$. z can be anywhere along the ray inclined at $\frac{-\pi}{4}$.



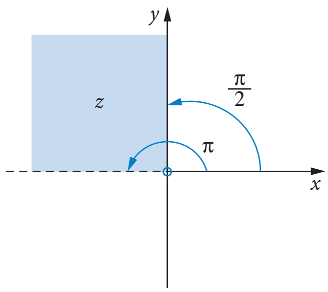
Example 17

Sketch the following subsets for the variable complex number z.

a $\frac{\pi}{2} \leq \arg(z) < \pi$ b $\arg(z) = \arg\left(1-i\sqrt{3}\right)$ c $\arg(z) + \arg\left(\sqrt{2}+i\sqrt{2}\right) = 0$

Solution

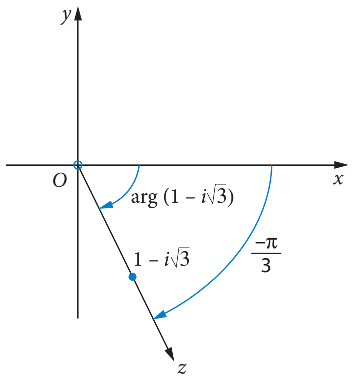
a z can be any point such that the angle between the vector z and the positive x-axis is at least $\frac{\pi}{2}$ but less than π. Note that the x-axis is dotted since $\arg(z) \neq \pi$.



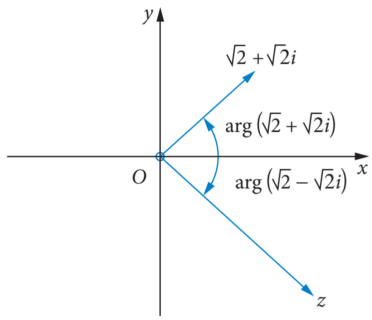
b The vector z and the vector to $1-i\sqrt{3}$ must be running in the same direction.

$\arg\left(1-i\sqrt{3}\right) = \frac{-\pi}{3}$

The locus of z is the ray from the origin with $\arg(z) = \frac{-\pi}{3}$.



c $\arg(z)+\arg\left(\sqrt{2}+i\sqrt{2}\right)=0$ can be rearranged to give $\arg(z)=-\arg\left(\sqrt{2}+i\sqrt{2}\right)$. The vector z lies along the ray on which the complex conjugate of $\left(\sqrt{2}+i\sqrt{2}\right)$, i.e. $\left(\sqrt{2}-i\sqrt{2}\right)$, lies.



Example 18

Sketch the locus of z if

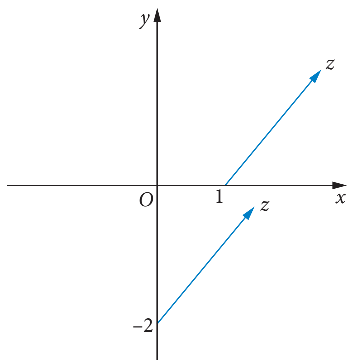
a $\arg(z-1)=\arg(z+2i)$

b $\arg(z-2)-\arg(z-i)=\pi$

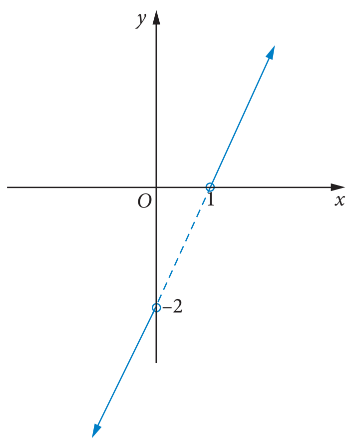
c $\arg(z-1)-\arg(z+3)=\dfrac{\pi}{2}$

Solution

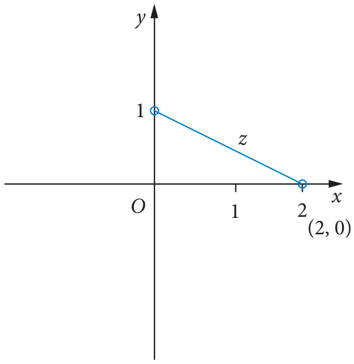
a The vector $z-1=z-(1+0i)$ and the vector $z+2i=z-(0-2i)$. z must be located so that the vectors are parallel (on the same line), going in the same direction.



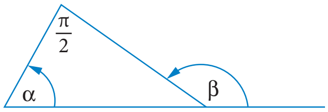
z can be anywhere on the two rays on either side of the interval between $(1, 0)$ and $(0, -2)$. The axis points are not included.



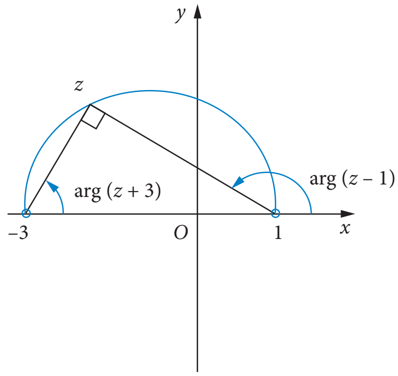
b In this case, the vectors $(z-2)$ and $(z-i)$ must be running in opposite directions. z is the set of points on the line interval joining (2, 0) and (0, 1), excluding the points on the axes.



c In this case, it is convenient to use the exterior angle theorem and the angle in a semicircle theorem as the difference between the arguments is a constant angle, i.e., $\beta-\alpha=\frac{\pi}{2}$.



So z is the set of points that lie on the semicircle with its diameter between the points (1, 0) and (−3, 0).



Since the argument of 0 is not defined, the terminating points of the subsets shown in this section are not included in the subsets. This can be emphasised as shown in Example **18** with an open circle at these points, although this is sometimes omitted.

WS
Complex plane graphs

EXERCISE 4.06 Graphing subsets of the complex plane: subsets not involving modulus

Concepts and techniques

1 Example 16 Sketch the following subsets of the complex plane.

a $\arg(z)=\frac{\pi}{3}$ b $\arg(z)=\frac{3\pi}{4}$ c $\arg(z)=\frac{-2\pi}{3}$ d $\arg(z)=\pi$

2 Sketch z if

a $\arg(z-1)=\frac{2\pi}{3}$ b $\arg(z-i)=\frac{\pi}{6}$ c $\arg(z-(1-i))=\frac{5\pi}{6}$ d $\arg(z-2-3i)=\frac{\pi}{4}$

3 Example 17 Sketch the following regions.

a $\frac{-\pi}{4}\le\arg(z)\le\frac{\pi}{4}$ b $\frac{\pi}{6}<\arg(z)<\frac{2\pi}{3}$ c $-\pi<\arg(z)<\frac{\pi}{2}$

4 Sketch each subset.

a $\arg(z)=\arg(1+i)$ b $\arg(z)=\arg\left(\sqrt{3}+i\right)$

c $\arg(z-1)=\arg\left(\sqrt{2}-i\sqrt{2}\right)$ d $\arg(z+2i)=\arg\left(-1+i\sqrt{3}\right)$

5 Sketch z if

a $\arg(z)-\arg(-1-i)=0$ b $\arg(z)-\arg\left(1+i\sqrt{3}\right)=0$

c $\arg(z)+\arg\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)=0$ d $\arg(z)+\arg\left(-\sqrt{3}-i\right)=0$

6 Sketch the following.

a $\arg(z-1)=\arg\left(-\sqrt{2}+i\sqrt{2}\right)$ b $\arg(z-i)=\arg\left(1+i\sqrt{3}\right)$

7 Example 18 Find the locus of z if:

a $\arg(z)=\arg(z-3)$ b $\arg(z)=\arg(z-2i)$

c $\arg(z+1)-\arg(z)=0$ d $\arg(z+i)-\arg(z-1)=0$

8 Sketch each of the following.

a $\arg(z-1)-\arg(z)=\pi$ b $\arg(z-2)-\arg(z+2)=\frac{\pi}{2}$

c $\arg(z-i)-\arg(z+3)=\frac{\pi}{2}$ d $\arg(z+i)-\arg(z-1)=\pi$

Reasoning and communication

9 Using the theorem $\arg\left(\frac{z}{w}\right)=\arg(z)-\arg(w)$, sketch each of the following.

a $\arg\left(\frac{z}{z-2}\right)=0$ b $\arg\left(\frac{z+1}{z-1}\right)=\frac{\pi}{2}$

c $\arg\left(\frac{z-1}{z-i}\right)=\pi$ d $\arg\left(\frac{z-2i}{z+2i}\right)=\frac{\pi}{2}$

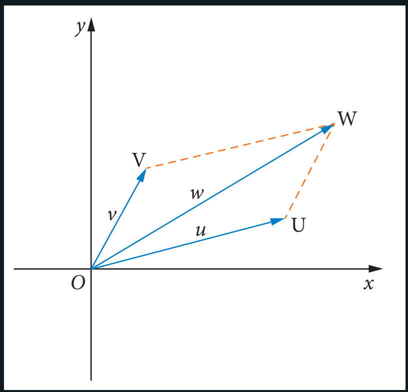
10 Describe each subset of the Argand plane defined by the equations below. Draw a diagram.

a $\arg\left(\frac{z-1}{z+i}\right)=0$ b $\arg\left(\frac{z-2}{z+2}\right)=\frac{\pi}{4}$

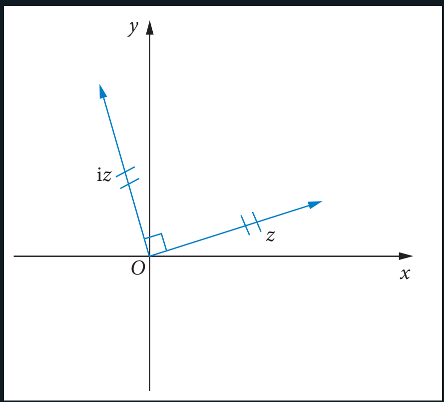
c $\arg\left(\frac{z+1}{z-1}\right)=\frac{\pi}{3}$ d $\arg\left(\frac{z-2i}{z+2}\right)=\frac{\pi}{2}$

CHAPTER SUMMARY
THE COMPLEX PLANE

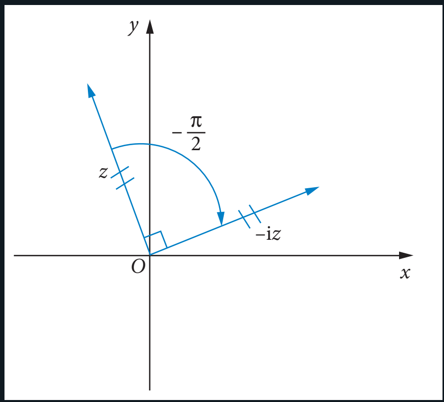
- The parallelogram rule for vectors: $\mathbf{w} = \mathbf{u} + \mathbf{v}$

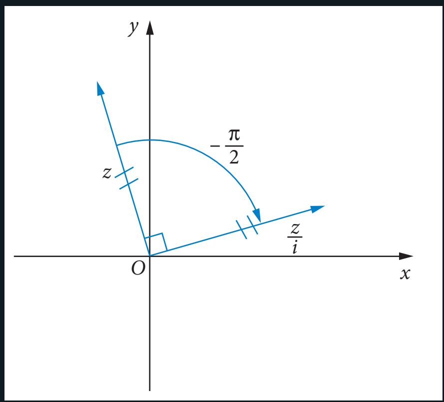


- If $z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)]$ are two complex numbers, then their **product** is $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$.
  - Multiplication by i is equivalent to a rotation of $\frac{\pi}{2}$ in the complex plane.



  - Multiplication by $-i$ or division by i are transformations equivalent to a rotation of $-\frac{\pi}{2}$ in the complex plane.



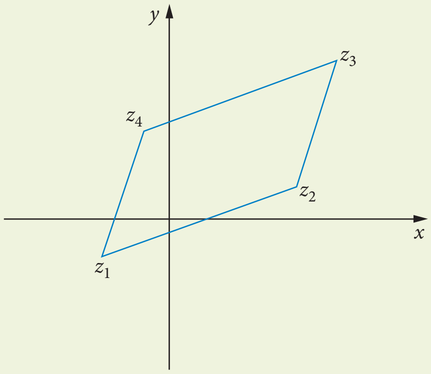


- If $z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)]$ are two complex numbers, then their **quotient** is $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$.
- For two non-zero complex numbers z_1 and z_2, $z_1 = z_2$ *if and only if* $|z_1| = |z_2|$ AND $\arg(z_1) = \arg(z_2)$.
- Subsets of the complex plane are sets of points representing complex numbers that are described by a certain rule or condition placed on the variable complex number z. The set of points is often called the **locus of** z. There are two approaches to finding the locus of z: *algebraic* and *geometric*.

CHAPTER REVIEW
THE COMPLEX PLANE

Multiple choice

1 Example 1 In the diagram, $PQRS$ is a parallelogram whose vertices represent the complex numbers z_1, z_2, z_3, z_4.

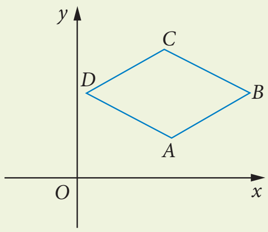


Which statement is true?

A $z_3 = z_4 + z_2$
B $z_3 - z_4 = z_2 - z_1$
C $|z_4 - z_1| = |z_3 - z_4|$
D $z_3 - z_1 = i(z_2 - z_4)$
E $\arg(z_3) = \arg(z_1)$

2 Example 2 In the diagram, $ABCD$ is a rhombus. The points A, B, C, D represent complex numbers a, b, c, d. The vertex C represents the complex number:

A $d - a + b - a$
B $a + b - d$
C $d - a + b$
D $c - d + c - b$
E $d - a + c - b$

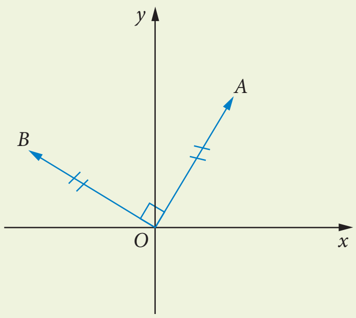


3 Example 3 The points O, A, B, C form a parallelogram $OABC$, where A represents $-1 + 2i$ and B represents $1 + 5i$. The point C is represented by

A $7i$
B $-2 - 3i$
C $3i$
D $2 + 3i$
E $-2 + 3i$

4 Example 4 Consider the two points A and B representing the complex numbers z and w respectively, where $\angle AOB = 90°$. Which of the following is true?

A $z = iw$
B $z = -iw$
C $z = w^2$
D $z = \frac{\pi}{2}w$
E $z = i^2w$



9780170250306

5 Example 5 Consider the two points P and Q representing the complex numbers z and w respectively, where $\arg(z) = \alpha$, $\arg(w) = 2\alpha$ and $|z| = |w| = 1$. Which of the following is true?

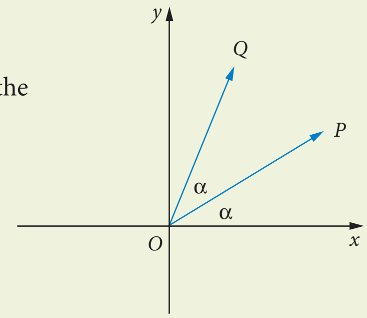
A $z = w^2$

B $\arg(w - z) = \alpha$

C $\arg(w + z) = 3\alpha$

D $\arg\left(\frac{z}{w}\right) = \alpha$

E $w = z^2$



Short answer

6 Example 6 If $z = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$, sketch z^2 on an Argand plane.

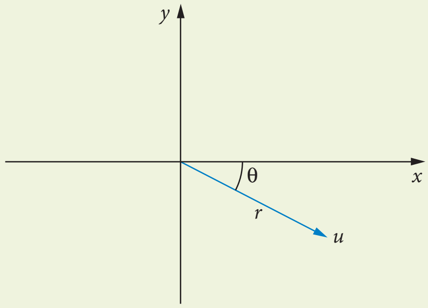
7 Example 7 Consider the complex numbers $z = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$ and $w = 4\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$.

a Explain why zw is purely real.

b Explain why $\frac{z}{w}$ is purely imaginary.

c Sketch zw and $\frac{z}{w}$ on an Argand diagram.

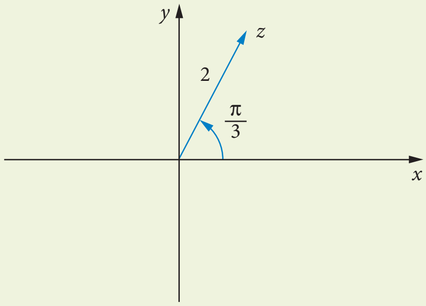
8 Example 8 The complex number u is sketched below. Sketch $\frac{u}{i}$ on the same diagram.



9 Example 9 The complex number $z = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$ is sketched below.

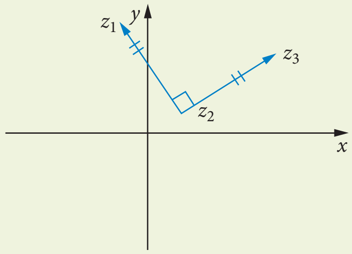
a State $\mathrm{mod}\left(\frac{1}{z}\right)$ and $\arg\left(\frac{1}{z}\right)$.

b Sketch z and $\frac{1}{z}$ on the same diagram.



9780170250306

10 Example 10 Consider the complex numbers z_1, z_2, z_3. Prove that $(z_3 - z_2)^2 + (z_1 - z_2)^2 = 0$.

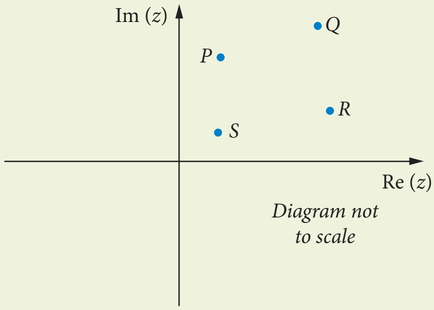


11 Example 11 The points P, Q, R, S representing the complex numbers z_1, z_2, z_3, z_4 form a quadrilateral. State what type of quadrilateral $PQRS$ is if

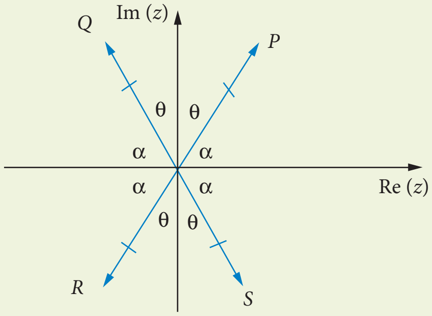
a $|z_3 - z_1| = |z_2 - z_4|$

b $z_3 - z_2 = z_4 - z_1$

c $\arg(z_3 - z_2) = \arg(z_4 - z_1)$



12 Example 12 Consider the points P, Q, R, S representing the complex numbers z_1, z_2, z_3, z_4 respectively. Express z_2, z_3, z_4 in terms of z_1.



13 Example 13 Sketch the locus of z and write the Cartesian equation or inequality for each of the following.

a $|z| = 3$ b $|z + 1| \leq 4$ c $\frac{1}{2} < |z - 1 - i| \leq 1$

14 Example 16 Sketch the locus of z on an Argand plane for each case below.

a $\arg(z) = \frac{2\pi}{3}$ b $\arg(z - i) = -\frac{\pi}{4}$ c $\arg(z + 1 + i) = \frac{3\pi}{4}$

9780170250306

15 Example 17 Sketch each subset of the complex plane for z defined by:

a $-\frac{\pi}{3} \leq \arg(z) < \frac{\pi}{3}$ b $\arg(z) = \arg(1+i)$ c $\arg(z) = \arg\left(z - \sqrt{3} + i\right)$

16 Example 18 Sketch each subset of the complex plane for z if:

a $\arg(z) = \arg(z-2)$

b $\arg(z-1) - \arg(z+i) = \pi$

c $\arg(z-2i) - \arg(z-i) = \frac{\pi}{6}$

Application

17 Sketch and describe the locus of z for each of the following.

a $|z-2| = \text{Re}(z)$ b $|z+6i| = -\text{Im}(z)$ c $\text{Re}(z) + \text{Im}(z) = -1$

18 Sketch the locus of z for each of the following.

a $-2 < \text{Re}(z) \leq 4$ b $|z+i| = |z-1|$ c $|\text{Re}(z)| \leq 2$ and $\frac{\pi}{4} < \arg(z) \leq \frac{3\pi}{4}$

Practice quiz